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Higher Auslander algebras admitting trivial maximal orthogonal subcategories

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ABSTRACT

In this paper, we prove that Λ is an $(n-1)$-Auslander Artinian algebra with $\mathrm{gl.dim}\,\Lambda = n$ $(\geqslant 2)$ admitting a trivial maximal $(n-1)$-orthogonal subcategory of $\operatorname{mod}\Lambda$ if and only if it is Morita equivalent to a finite product of F and $\begin{pmatrix} F & F & 0 & \cdots & 0 & 0 \\ 0 & F & F & \cdots & 0 & 0 \\ 0 & 0 & F & \cdots & 0 & 0 \\ & \cdots & & & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & F & F \\ 0 & 0 & 0 & \cdots & 0 & F \end{pmatrix}_{(n+1)\times(n+1)}$, where F is a division algebra. In addition, we obtain a necessary condition for an Auslander Artinian algebra admitting a non-trivial maximal 1-orthogonal subcategory.

1. Introduction

It is well known that the notion of maximal n-orthogonal subcategories introduced by Iyama in [Iy3] played a crucial role in developing the higher-dimensional Auslander–Reiten theory (see [Iy3] and [Iy4]). This notion coincides with that of $(n+1)$-cluster tilting subcategories introduced by Keller and Reiten in [KR]. In general, maximal n-orthogonal subcategories rarely exist. So it would be interesting to investigate when maximal n-orthogonal subcategories exist and the properties of algebras admitting such subcategories. Several authors have worked on this topic (see [EH,GLS,HuZ,Iy3,Iy4,Iy5, Iy6,L], and so on). As a generalization of the notion of the classical Auslander algebras, Iyama in-

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troduced the notion of n-Auslander algebras in [Iy6]. Then he proved that for an $(n-1)$-Auslander Artinian algebra Λ with global dimension n $(\geqslant 2)$, Λ has maximal $(n-1)$-orthogonal modules in mod Λ if and only if Λ is Morita equivalent to $T_m^{(n)}(F)$ for some $m \geqslant 1$, where F is a division algebra, $T_m^{(1)}(F)$ is an $m \times m$ upper triangular matrix algebra and $T_m^{(n)}(F)$ is the endomorphism algebra of a maximal $(n-2)$-orthogonal module in mod $T_m^{(n-1)}(F)$. Moreover, he gave some examples of the quivers of these algebras inductively. In [HuZ] we proved that an $(n-1)$-Auslander Artinian algebra Λ with global dimension n $(\geqslant 2)$ admits a trivial maximal $(n-1)$-orthogonal subcategory of mod Λ if and only if any simple module $S \in \operatorname{mod} \Lambda$ with projective dimension n is injective. In [HuZ] we also proved that for an almost hereditary Artinian algebra Λ with global dimension 2, if Λ admits a maximal 1-orthogonal subcategory $\mathscr{C}$ of mod Λ, then $\mathscr{C}$ is trivial. In this paper, we continue to study the structure of an $(n-1)$-Auslander Artinian algebra Λ admitting a trivial maximal $(n-1)$-orthogonal subcategory of mod Λ. This paper is organized as follows.

In Section 2, we give some notions and notations and collect some preliminary results about minimal morphisms. In Section 3, we give some homological properties of indecomposable modules (in particular, simple modules) over higher Auslander Artinian algebras (admitting a trivial maximal orthogonal subcategory of mod Λ).

Let Λ be an $(n-1)$-Auslander Artinian algebra with global dimension n $(\geqslant 2)$ admitting a trivial maximal $(n-1)$-orthogonal subcategory of mod Λ. In Section 4, we first prove the following results: (1) For any indecomposable non-projective–injective module $M \in \operatorname{mod} \Lambda$ with projective dimension $n-i$ and $0 \leqslant i \leqslant n$, there exists a simple module $S \in \operatorname{mod} \Lambda$ with projective dimension n such that M is isomorphic to the ith syzygy of S. (2) For any simple module $S \in \operatorname{mod} \Lambda$ with projective dimension n, the ith syzygy of S is simple for any $1 \leqslant i \leqslant n$ and all terms in a minimal projective resolution of S are indecomposable. By using these results, we then prove that Λ is Morita equivalent to a finite product of F and $T_{n+1}(F)/J^2(T_{n+1}(F))$, where F is a division algebra, $T_{n+1}(F)$ is an $(n+1) \times (n+1)$ upper triangular matrix algebra over F and $J(T_{n+1}(F))$ is the Jacobson radical of $T_{n+1}(F)$. We remark that this algebra is $T_2^{(n)}(F)$ in Iyama's result mentioned above.

By [Iy6], there exists an Auslander Artinian algebra with global dimension 2 admitting a non-trivial maximal 1-orthogonal subcategory. On the other hand, by [HuZ, Corollary 3.12] we have that if Λ is an Auslander Artinian algebra with global dimension 2 admitting a non-trivial maximal 1-orthogonal subcategory of mod Λ, then there exists a simple module $S \in \operatorname{mod} \Lambda$ such that both the projective and injective dimensions of S are equal to 2. In Section 5, we further give some necessary condition for an Auslander Artinian algebra with global dimension 2 admitting a non-trivial maximal 1-orthogonal subcategory in terms of the homological properties of simple modules. We prove that if Λ is an Auslander Artinian algebra with global dimension 2 admitting a non-trivial maximal 1-orthogonal subcategory of mod Λ, then there exist at least two non-injective simple modules in mod Λ with projective dimension 2. Some examples are given to illustrate this result.

2. The properties of minimal morphisms

In this section, we give some notions and notations in our terminology and collect some preliminary results about minimal morphisms for later use.

Throughout this paper, Λ is an Artinian algebra with the center R, mod Λ is the category of finitely generated left Λ-modules and $\operatorname{gl.dim} \Lambda$ denotes the global dimension of Λ. We denote by $\mathbb{D}$ the ordinary duality between mod Λ and mod Λ^{op}, that is, $\mathbb{D}(-) = \operatorname{Hom}_R(-, I(R/J(R)))$, where $J(R)$ is the Jacobson radical of R and $I(R/J(R))$ is the injective envelope of $R/J(R)$.

Let M be in mod Λ. We use

$$\cdots \to P_i(M) \to \cdots \to P_1(M) \to P_0(M) \to M \to 0$$

and

$$0 \to M \to I^0(M) \to I^1(M) \to \cdots \to I^i(M) \to \cdots$$

to denote a minimal projective resolution and a minimal injective resolution of M, respectively. In particular, $P_0(M)$ and $I^0(M)$ are the projective cover and the injective envelope of M, respectively. Denote by $\Omega^i M$ and $\Omega^{-i} M$ the ith syzygy and ith cosyzygy of M, respectively.

The following easy observations are well known.

Lemma 2.1. *Let $M \in \operatorname{mod} \Lambda$ and $M \cong M_1 \oplus M_2$. Then*

$$\cdots \to P_1(M') \oplus P_1(M'') \to P_0(M') \oplus P_0(M'') \to M(\cong M_1 \oplus M_2) \to 0$$

and

$$0 \to M(\cong M_1 \oplus M_2) \to I^0(M') \oplus I^0(M'') \to I^1(M') \oplus I^1(M'') \to \cdots$$

are a minimal projective resolution and a minimal injective resolution of M, respectively, and $\Omega^i M \cong \Omega^i M_1 \oplus \Omega^i M_2$ and $\Omega^{-i} M \cong \Omega^{-i} M_1 \oplus \Omega^{-i} M_2$ for any $i \geqslant 1$.

Lemma 2.2. *Let M and S be in $\operatorname{mod} \Lambda$ with S simple. Then $\operatorname{Ext}^i_\Lambda(S, M) \cong \operatorname{Hom}_\Lambda(S, \Omega^{-i} M)$ for any $i \geqslant 0$.*

Recall from [AuR] that a morphism $f : M \to N$ in $\operatorname{mod} \Lambda$ is said to be *left minimal* if an endomorphism $g : N \to N$ is an automorphism whenever $f = gf$. Dually, the notion of *right minimal morphisms* is defined.

Lemma 2.3. *(See [Au, Chapter II, Lemma 4.3].) Let $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ be a non-split exact sequence in* $\operatorname{mod} \Lambda$.

(1) *If A is indecomposable, then $f : B \to C$ is right minimal.*
(2) *If C is indecomposable, then $g : A \to B$ is left minimal.*

By Lemma 2.3, we immediately have the following result.

Corollary 2.4. *Let $M \in \operatorname{mod} \Lambda$ be an indecomposable non-injective module and $I^0(M)$ projective. Then*

$$\cdots \to P_i(M) \to \cdots \to P_1(M) \to P_0(M) \to I^0(M) \xrightarrow{\pi} I^0(M)/M \to 0$$

is a minimal projective resolution of $I^0(M)/M$, where π is the natural epimorphism.

The following properties of minimal morphisms are useful in the rest of the paper.

Lemma 2.5. *Let $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ be a non-split exact sequence in* $\operatorname{mod} \Lambda$.

(1) *If g is left minimal, then $\operatorname{Ext}^1_\Lambda(C', A) \neq 0$ for any non-zero direct summand C' of C.*
(2) *If f is right minimal, then $\operatorname{Ext}^1_\Lambda(C, A') \neq 0$ for any non-zero direct summand A' of A.*

Proof. (1) If $\operatorname{Ext}^1_\Lambda(C', A) = 0$ holds true for some non-zero direct summand C' of C, then we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & C' \oplus A \arrow[r, shift left, "\pi_3"] \arrow[d, "i_1"] & C' \arrow[r] \arrow[l, dashed, shift left, "i_3"] \arrow[d, shift left, "i_2"] & 0 \\
0 \arrow[r] & A \arrow[r, "g"] & B \arrow[r, "f"] & C \arrow[r] \arrow[u, dashed, shift left, "\pi_2"] & 0
\end{tikzcd}$$

such that $\pi_3 i_3 = 1_{C'} = \pi_2 i_2$ and $i_2\pi_3 = f i_1$. Then $1_{C'} = (\pi_2 i_2)(\pi_3 i_3) = (\pi_2 f)(i_1 i_3)$, and hence C' is a direct summand of B and $(\pi_2 f)g = 0$. By [AuRS, Chapter I, Theorem 2.4], g is not left minimal, which is a contradiction.

Dually, we get (2). □

The following lemma establishes a connection between left minimal morphisms and right minimal morphisms.

Lemma 2.6. *Let*

$$0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0 \tag{1}$$

be a non-split exact sequence in mod Λ *with* B *projective–injective. Then the following statements are equivalent.*

(1) *A is indecomposable and g is left minimal.*
(2) *C is indecomposable and f is right minimal.*

Proof. (1) ⇒ (2) Since A is indecomposable, f is right minimal by Lemma 2.3. Notice that B is projective by assumption, so the exact sequence (1) is part of a minimal projective resolution of C. If $C = C_1 \oplus C_2$ with C_1 and C_2 non-zero, then neither C_1 nor C_2 is projective by Lemma 2.5. So both $\Omega^1 C_1$ and $\Omega^1 C_2$ are non-zero and $A \cong \Omega^1 C_1 \oplus \Omega^1 C_2$, which contradicts the fact that A is indecomposable.

Similarly, we get (2) ⇒ (1). □

3. Higher Auslander algebras and maximal orthogonal subcategories

In this section, we give the definitions of higher Auslander algebras and maximal orthogonal subcategories, which were introduced by Iyama in [Iy6] and [Iy3], respectively. Then we study the homological behavior of indecomposable modules (in particular, simple modules) over higher Auslander algebras (admitting a trivial maximal orthogonal subcategory of mod Λ).

As a generalization of the notion of classical Auslander algebras, Iyama introduced in [Iy6] the notion of n-Auslander algebras as follows.

Definition 3.1. (See [Iy6].) For a positive integer n, Λ is called an *n-Auslander algebra* if $\operatorname{gl.dim} \Lambda \leqslant n+1$ and $I^0(\Lambda), I^1(\Lambda), \dots, I^n(\Lambda)$ are projective.

The notion of n-Auslander algebras is left–right symmetric by [Iy6, Theorem 1.10]. It is trivial that n-Auslander algebras with global dimension at most n are semisimple. In particular, the notion of 1-Auslander algebras is just that of classical Auslander algebras. In the following, we assume that $n \geqslant 2$ when an $(n-1)$-Auslander algebra is concerned.

Denote by $\mathscr{P}\mathscr{I}^n(\Lambda)$ (resp. $\mathscr{I}\mathscr{P}^n(\Lambda)$) the subcategory of mod Λ consisting of indecomposable projective modules with injective dimension n (resp. indecomposable injective modules with projective dimension n). By applying Lemma 2.6 to $(n-1)$-Auslander algebras, we get the following result.

Lemma 3.2. *Let Λ be an $(n-1)$-Auslander algebra with* $\operatorname{gl.dim} \Lambda = n$. *Then we have the following*:

(1) *For any $P \in \mathscr{P}\mathscr{I}^n(\Lambda)$, the minimal injective resolution of P*

$$0 \to P \to I^0(P) \to I^1(P) \to \cdots \to I^n(P) \to 0 \tag{2}$$

is a minimal projective resolution of $I^n(P)$ and $I^n(P)$ is indecomposable.

(2) *For any module $I \in \mathscr{I}\mathscr{P}^n(\Lambda)$, the minimal projective resolution of I*

$$0 \to P_n(I) \to \cdots \to P_1(I) \to P_0(I) \to I \to 0$$

is a minimal injective resolution of $P_n(I)$ and $P_n(I)$ is indecomposable.

Proof. (1) Since Λ is an $(n-1)$-Auslander algebra, by Lemma 2.1 it is easy to see that $I^i(P)$ is projective for any $0 \leqslant i \leqslant n-1$. So the exact sequence (2) is a projective resolution of $I^n(P)$, and then the assertion follows from Lemma 2.6.

Dually, we get (2). $\square$

By Lemma 3.2, we get immediately the following result.

Lemma 3.3. *Let Λ be an $(n-1)$-Auslander algebra with* $\operatorname{gl.dim}\Lambda = n$. *Then Ω^n gives a one–one correspondence between $\mathscr{I}\mathscr{P}^n(\Lambda)$ and $\mathscr{P}\mathscr{I}^n(\Lambda)$ with the inverse Ω^{-n}.*

For a module $M \in \operatorname{mod}\Lambda$, we use $\operatorname{pd}_\Lambda M$ and $\operatorname{id}_\Lambda M$ to denote the projective dimension and the injective dimension of M, respectively.

Lemma 3.4. *Let Λ be an $(n-1)$-Auslander algebra with* $\operatorname{gl.dim}\Lambda = n$ *and let $S \in \operatorname{mod}\Lambda$ be a simple module with* $\operatorname{pd}_\Lambda S = n$. *Then $P_n(S)$ is indecomposable.*

Proof. Let Λ be an $(n-1)$-Auslander algebra with $\operatorname{gl.dim}\Lambda = n$ and let $S \in \operatorname{mod}\Lambda$ be a simple module with $\operatorname{pd}_\Lambda S = n$. By [Iy2, Proposition 6.3(2)], $\operatorname{Ext}^n_\Lambda(S, \Lambda) \in \operatorname{mod}\Lambda^{op}$ is simple. By [HuZ, Lemma 2.4], $S \nsubseteq I^0(\Lambda) \oplus \cdots \oplus I^{n-1}(\Lambda)$. So $\operatorname{Ext}^i_\Lambda(S, \Lambda) \cong \operatorname{Hom}_\Lambda(S, I^i(\Lambda)) = 0$ for any $0 \leqslant i \leqslant n-1$ by Lemma 2.2. Then from the minimal projective resolution of S, we get the exact sequence:

$$0 \to P_0(S)^* \to \cdots \to P_{n-1}(S)^* \to P_n(S)^* \to \operatorname{Ext}^n_\Lambda(S, \Lambda) \to 0$$

which is a minimal projective resolution of $\operatorname{Ext}^n_\Lambda(S, \Lambda)$ by [M, Proposition 4.2], where $(-)^* = \operatorname{Hom}_\Lambda(-, \Lambda)$. So $P_n(S)^* \cong P_0(\operatorname{Ext}^n_\Lambda(S, \Lambda))$ is indecomposable and hence $P_n(S)$ is also indecomposable. $\square$

Denote by $\mathscr{P}^n(S)$ and $\mathscr{I}^n(S)$ the subcategory of $\operatorname{mod}\Lambda$ consisting of simple modules with projective dimension n and injective dimension n, respectively. Since $\mathbb{D}$ is a duality between simple Λ-modules and simple Λ^{op}-modules, we get easily the following result from [Iy2, Proposition 6.3].

Lemma 3.5. *Let Λ be an $(n-1)$-Auslander algebra with* $\operatorname{gl.dim}\Lambda = n$. *Then the functor* $\mathbb{D}\operatorname{Ext}^n_\Lambda(-, \Lambda)$ *gives a bijection from $\mathscr{P}^n(S)$ to $\mathscr{I}^n(S)$ with the inverse* $\operatorname{Ext}^n_\Lambda(-, \Lambda)\mathbb{D}$.

Let $\mathscr{C}$ be a full subcategory of $\operatorname{mod}\Lambda$ and let n be a positive integer. Recall from [AuR] that $\mathscr{C}$ is said to be *contravariantly finite* in $\operatorname{mod}\Lambda$ if for any $M \in \operatorname{mod}\Lambda$, there exists a morphism $C_M \to M$ with $C_M \in \mathscr{C}$ such that $\operatorname{Hom}_\Lambda(C, C_M) \to \operatorname{Hom}_\Lambda(C, M) \to 0$ is exact for any $C \in \mathscr{C}$. Dually, the notion of *covariantly finite subcategories* of $\operatorname{mod}\Lambda$ is defined. A full subcategory of $\operatorname{mod}\Lambda$ is said to be *functorially finite* in $\operatorname{mod}\Lambda$ if it is both contravariantly finite and covariantly finite in $\operatorname{mod}\Lambda$. We denote by ${}^{\perp_n}\mathscr{C} = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X, C) = 0$ for any $C \in \mathscr{C}$ and $1 \leqslant i \leqslant n\}$, and $\mathscr{C}^{\perp_n} = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(C, X) = 0$ for any $C \in \mathscr{C}$ and $1 \leqslant i \leqslant n\}$.

Definition 3.6. (See [Iy3].) Let $\mathscr{C}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$. For a positive integer n, $\mathscr{C}$ is called a *maximal n-orthogonal subcategory* of $\operatorname{mod}\Lambda$ if $\mathscr{C} = {}^{\perp_n}\mathscr{C} = \mathscr{C}^{\perp_n}$.

For a module $M \in \operatorname{mod} \Lambda$, we use $\operatorname{add}_\Lambda M$ to denote the subcategory of mod Λ consisting of all modules isomorphic to direct summands of finite direct sums of copies of ${}_\Lambda M$. From the definition above, we get easily that both Λ and $\mathbb{D}\Lambda^{op}$ are in any maximal n-orthogonal subcategory of mod Λ. So $\operatorname{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op})$ is contained in any maximal n-orthogonal subcategory of mod Λ. On the other hand, it is easy to see that if $\operatorname{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op})$ is a maximal n-orthogonal subcategory of mod Λ, then $\operatorname{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op})$ is the unique maximal n-orthogonal subcategory of mod Λ. In this case, we say that Λ admits a *trivial maximal n-orthogonal subcategory* of mod Λ (see [HuZ]).

For a positive integer n, we proved in [HuZ, Proposition 3.2] that Λ admits no maximal j-orthogonal subcategories of mod Λ for any $j \geqslant n$ if $\operatorname{id}_\Lambda \Lambda = n$ (especially, if $\operatorname{gl.dim} \Lambda = n$). Furthermore, in [HuZ] we obtained an equivalent characterization for the existence of the trivial maximal $(n-1)$-orthogonal subcategory of mod Λ over an $(n-1)$-Auslander algebra Λ with $\operatorname{gl.dim} \Lambda = n$ as follows.

Lemma 3.7. *(See [HuZ, Corollary 3.10].) Let Λ be an $(n-1)$-Auslander algebra with* $\operatorname{gl.dim} \Lambda = n$. *Then the following statements are equivalent.*

(1) *Λ admits a trivial maximal $(n-1)$-orthogonal subcategory* $\operatorname{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op})$ *of* mod Λ.
(2) *A simple module $S \in \operatorname{mod} \Lambda$ is injective if $\operatorname{pd}_\Lambda S = n$.*

For a positive integer n, recall from [FGR] that Λ is called *n-Gorenstein* if $\operatorname{pd}_\Lambda I^i(\Lambda) \leqslant i$ for any $0 \leqslant i \leqslant n-1$. By [FGR, Theorem 3.7], the notion of n-Gorenstein algebras is left–right symmetric. Recall from [B] that Λ is called *Auslander–Gorenstein* if Λ is n-Gorenstein for all n and both $\operatorname{id}_\Lambda \Lambda$ and $\operatorname{id}_{\Lambda^{op}} \Lambda$ are finite.

Lemma 3.8. *Assume that* $\operatorname{id}_\Lambda \Lambda = \operatorname{id}_{\Lambda^{op}} \Lambda = n$ $(< \infty)$. *Then we have the following*:

(1) ([IS, Proposition 1(1)]) $\operatorname{pd}_\Lambda X = n$ *or* ∞ *for any non-zero submodule X of $I^n(\Lambda)$.*
(2) ([IS, Corollary 7(2)]) *If Λ is Auslander–Gorenstein and $I \in \mathscr{I}\mathscr{P}^n(\Lambda)$, then $I \cong I^0(S)$ for some simple module $S \in \operatorname{mod} \Lambda$ with $\operatorname{pd}_\Lambda S = n$ or ∞.*

For a module $M \in \operatorname{mod} \Lambda$, the *grade* of M, denoted by $\operatorname{grade} M$, is defined as $\inf\{n \geqslant 0 \mid \operatorname{Ext}^n_\Lambda(M, \Lambda) \neq 0\}$ (see [AuB]).

Lemma 3.9. *(See [Iy1, Proposition 2.4].) Let Λ be an n-Gorenstein algebra. Then the subcategory $\{X \in \operatorname{mod} \Lambda \mid \operatorname{grade} X \geqslant n\}$ of* mod Λ *is closed under submodules and factor modules.*

Lemma 3.10. *(See [HuZ, Lemma 3.4].) If* $\operatorname{gl.dim} \Lambda = n \geqslant 2$ *and $\mathscr{C}$ is a subcategory of* mod Λ *such that $\Lambda \in \mathscr{C}$ and $\operatorname{Ext}^j_\Lambda(\mathscr{C}, \mathscr{C}) = 0$ for any $1 \leqslant j \leqslant n-1$, then* $\operatorname{grade} M = n$ *for any $M \in \mathscr{C}$ without projective direct summands.*

4. The existence of trivial maximal orthogonal subcategories

In this section, by studying the properties of the syzygies and the terms in a minimal projective resolution of a simple module, we will give a complete classification of $(n-1)$-Auslander algebras with $\operatorname{gl.dim} \Lambda = n$ admitting a trivial maximal $(n-1)$-subcategory. The main result has a strong relationship with Iyama's classification of $(n-1)$-Auslander algebras admitting n-cluster tilting modules in [Iy6].

We begin with the following

Lemma 4.1. *Let Λ be an $(n-1)$-Auslander algebra with* $\operatorname{gl.dim} \Lambda = n$ *and let $S \in \operatorname{mod} \Lambda$ be a simple module. Then we have*:

(1) *If* $\mathrm{pd}_\Lambda S \leqslant n-1$, *then* $I^0(S)$ *is projective.*
(2) *If* $\mathrm{pd}_\Lambda S = n$, *then* $\mathrm{pd}_\Lambda I^0(S) = n$.

Proof. For any $0 \leqslant i \leqslant n$, if $\mathrm{pd}_\Lambda S = i$, then $\mathrm{Hom}_\Lambda(S, I^i(\Lambda)) \cong \mathrm{Ext}^i_\Lambda(S, \Lambda) \neq 0$. It follows that $I^0(S)$ is isomorphic to a direct summand of $I^i(\Lambda)$. Because Λ is an $(n-1)$-Auslander algebra, (1) follows trivially, and (2) follows from Lemma 3.8(1). □

For a module $M \in \mathrm{mod}\ \Lambda$, we use $\text{Ł}(M)$ to denote the length of M.

Lemma 4.2. *Let Λ be an $(n-1)$-Auslander algebra with* $\mathrm{gl.dim}\,\Lambda = n$ *admitting a trivial maximal $(n-1)$-orthogonal subcategory of* $\mathrm{mod}\ \Lambda$ *and let $M \in \mathrm{mod}\ \Lambda$ be indecomposable. If $\text{Ł}(M) \geqslant 2$ or M is not injective, then the following equivalent conditions hold true.*

(1) $\mathrm{pd}_\Lambda S \leqslant n-1$ *for any simple submodule S of M.*
(2) $I^0(M)$ *is projective.*

Proof. By Lemma 3.7, a simple module $S \in \mathrm{mod}\ \Lambda$ is injective if $\mathrm{pd}_\Lambda S = n$. Because $M \in \mathrm{mod}\ \Lambda$ is indecomposable, we have that $\mathrm{pd}_\Lambda S \leqslant n-1$ for any simple submodule S of M and the assertion (1) holds true. Otherwise, we have $M \cong S$, which contradicts the assumption that $\text{Ł}(M) \geqslant 2$ or M is not injective.

It suffices to prove (1) $\Rightarrow$ (2). By Lemma 4.1(1), it is easy to get the desired conclusion. □

The following proposition plays a crucial role in the proof of the main result in this paper.

Proposition 4.3. *Let Λ be an $(n-1)$-Auslander algebra with* $\mathrm{gl.dim}\,\Lambda = n$ *and $0 \leqslant i \leqslant n$. If Λ admits a trivial maximal $(n-1)$-orthogonal subcategory of* $\mathrm{mod}\ \Lambda$, *then for any indecomposable non-projective–injective module $M \in \mathrm{mod}\ \Lambda$ with $\mathrm{pd}_\Lambda M = n-i$, there exists a simple module $S \in \mathrm{mod}\ \Lambda$ such that $\mathrm{pd}_\Lambda S = n$ and $M \cong \Omega^i S$.*

Proof. For the case $i = 0$, it suffices to prove that $\text{Ł}(M) = 1$. Then M is simple and it is injective by Lemma 3.7. Thus the assertion follows.

Assume that $\text{Ł}(M) \geqslant 2$. By Lemma 4.2, $\mathrm{pd}_\Lambda S \leqslant n-1$ for any simple submodule S of M and $I^0(M)$ is projective.

If M is injective, then $M \cong I^0(S)$ for some simple Λ-module S with $\mathrm{pd}_\Lambda S = n$ by Lemma 3.8(2), which is a contradiction. Now assume that $\mathrm{id}_\Lambda M \geqslant 1$. By Corollary 2.4,

$$0 \to P_n(M) \to \cdots \to P_1(M) \to P_0(M) \to I^0(M) \xrightarrow{\pi} I^0(M)/M \to 0$$

is a minimal projective resolution of $I^0(M)/M$ and $\mathrm{pd}_\Lambda I^0(M)/M = n+1$, which contradicts the assumption that $\mathrm{gl.dim}\,\Lambda = n$. So the case for $i = 0$ is proved.

For the case $i = n$, we have that M is projective. Then M is not injective by assumption. Because $\mathrm{gl.dim}\,\Lambda = n$, $\mathrm{id}_\Lambda M \leqslant n$. On the other hand, because Λ admits a trivial maximal $(n-1)$-orthogonal subcategory of $\mathrm{mod}\ \Lambda$, $\mathrm{Ext}^j_\Lambda(\mathbb{D}\Lambda^{op}, \Lambda) = 0$ for any $1 \leqslant j \leqslant n-1$. Then it is not difficult to show that $\mathrm{id}_\Lambda M = n$. By Lemma 3.3, there exists an indecomposable injective module $T \in \mathrm{mod}\ \Lambda$ with $\mathrm{pd}_\Lambda T = n$ such that $M \cong \Omega^n T$. By the above argument, T is simple.

Now suppose $1 \leqslant i \leqslant n-1$. Then $\mathrm{pd}_\Lambda M = n-i \neq 0$. We claim that M is not injective. Otherwise, if M is injective, then the minimal projective resolution of M splits because $\mathrm{Ext}^j_\Lambda(\mathbb{D}\Lambda^{op}, \Lambda) = 0$ for any $1 \leqslant j \leqslant n-1$. It follows that M is projective, which is a contradiction. The claim is proved. Then by Lemma 4.2, $\mathrm{pd}_\Lambda S \leqslant n-1$ for any simple submodule S of M and $I^0(M)$ is projective. In the following, we will prove the assertion by induction on i.

If $i = 1$, then $\mathrm{pd}_\Lambda M = n-1$. By Lemma 2.6 and Corollary 2.4, $\mathrm{pd}_\Lambda I^0(M)/M = n$. So $I^0(M)/M \cong S$ for some simple module S with $\mathrm{pd}_\Lambda S = n$ by the above argument, and hence $M \cong \Omega^1 S$.

Assume that $2 \leqslant i \leqslant n-1$ and $\mathrm{pd}_\Lambda M = n-i$. By Corollary 2.4, we have a minimal projective resolution of $I^0(M)/M$ as follows.

$$0 \to P_{n-i}(M) \to \cdots \to P_1(M) \to P_0(M) \to I^0(M) \xrightarrow{\pi} I^0(M)/M \to 0.$$

Then $\mathrm{pd}_\Lambda I^0(M)/M = n-(i-1)$ and $I^0(M)/M$ is indecomposable by Lemma 2.6. By the induction hypothesis, $I^0(M)/M \cong \Omega^{i-1}S$ for some simple module $S \in \mathrm{mod}\ \Lambda$ with $\mathrm{pd}_\Lambda S = n$. It follows that $M \cong \Omega^i S$. $\square$

As a consequence of Proposition 4.3, we get the following

Proposition 4.4. *Let Λ be an $(n-1)$-Auslander algebra with* $\mathrm{gl.dim}\,\Lambda = n$ *and let $S \in \mathrm{mod}\ \Lambda$ be a simple module with* $\mathrm{pd}_\Lambda S = n$*. If Λ admits a trivial maximal $(n-1)$-orthogonal subcategory of* $\mathrm{mod}\ \Lambda$*, then $\Omega^i S$ is simple and $P_i(S)$ is indecomposable for any $0 \leqslant i \leqslant n$.*

Proof. Let $S \in \mathrm{mod}\ \Lambda$ be a simple module with $\mathrm{pd}_\Lambda S = n$. By Lemma 3.7, S is injective. It follows from Lemma 3.2(2) that the minimal projective resolution of S

$$0 \to P_n(S) \to \cdots \to P_1(S) \to P_0(S) \to S \to 0$$

is a minimal injective resolution of $P_n(S)$.

We proceed by induction on i. The case for $i = 0$ holds true trivially, and the case for $i = n$ follows from Lemma 3.4 and the dual version of Proposition 4.3.

Now assume that $1 \leqslant i \leqslant n-1$ and $S' \in \mathrm{mod}\ \Lambda$ is a simple submodule of $\Omega^i S$. Because Λ is an $(n-1)$-Auslander algebra and S is injective, $P_0(S)$ is projective–injective and indecomposable. So S' is the unique simple submodule of $P_0(S)$ and hence $I^0(S') = P_0(S)$. By Lemma 2.2, $\mathrm{Ext}_\Lambda^{n-1}(S', P_n(S)) \cong \mathrm{Hom}_\Lambda(S', \Omega^1 S) \neq 0$, which implies that $\mathrm{pd}_\Lambda S' \geqslant n-1$. Because $\mathrm{gl.dim}\,\Lambda = n$, it is easy to see that $\mathrm{pd}_\Lambda S' = n-1$. Then by Proposition 4.3, there exists a simple module $S \in \mathrm{mod}\ \Lambda$ such that $\mathrm{pd}_\Lambda S = n$ and $S' \cong \Omega^1 S$. By Lemma 3.7, S is injective. So $\mathrm{id}_\Lambda S' = 1$ by Lemma 3.2(2).

Connecting a minimal projective resolution and a minimal injective resolution of S', then by Lemma 2.6, the following exact sequence is a minimal projective resolution of $I^1(S')$:

$$0 \to P_{n-1}(S') \to \cdots \to P_0(S') \to I^0(S')\bigl(\cong P_0(S)\bigr) \to I^1(S') \to 0$$

with $I^1(S')$ indecomposable. So $\mathrm{pd}_\Lambda I^1(S') = n$ and hence $I^1(S')$ is simple by Lemma 3.2(1). It follows that $S \cong I^1(S')$ and $\Omega^1 S \cong S'$ is simple. Thus $P_1(S)$ is indecomposable. The case for $i = 1$ is proved.

Now suppose $2 \leqslant i \leqslant n-1$. By Lemma 2.2, $\mathrm{Ext}_\Lambda^{n-i}(S', P_n(S)) \cong \mathrm{Hom}_\Lambda(S', \Omega^i S) \neq 0$. So $\mathrm{pd}_\Lambda S' (= t) \geqslant n-i$.

Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & S' \arrow[r] \arrow[d, "\alpha"] & P_{i-1}(S) \arrow[r, "\pi"] \arrow[d, equal] & M \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & \Omega^i S \arrow[r] & P_{i-1}(S) \arrow[r] & \Omega^{i-1} S \arrow[r] & 0
\end{tikzcd}$$

where $M = P_{i-1}(S)/S'$, α is an embedding homomorphism and β is an induced homomorphism. By the induction hypothesis, $\Omega^{i-1}S$ is simple and hence $P_{i-1}(S)$ is indecomposable. Then, by Lemma 2.6, M is indecomposable and π is right minimal. It follows that $\mathrm{pd}_\Lambda M = t+1$. Thus $M \cong \Omega^{n-t-1}S''$ for some simple module $S'' \in \mathrm{mod}\ \Lambda$ with $\mathrm{pd}_\Lambda S'' = n$ by Proposition 4.3. Because $i \geqslant n-t-1$, M is simple by the induction hypothesis. It is clear that β is an epimorphism and so it is an isomorphism, which implies that α is an isomorphism and $\Omega^i S \cong S'$ is simple. It follows that $P_i(S)$ is indecomposable. $\square$

The following result is an immediate consequence of Propositions 4.3 and 4.4.

Corollary 4.5. *Let Λ be an $(n-1)$-Auslander algebra with* $\mathrm{gl.dim}\,\Lambda = n$ *admitting a trivial maximal $(n-1)$-orthogonal subcategory of* $\operatorname{mod}\Lambda$. *Then we have*:

(1) *Any indecomposable module $M \in \operatorname{mod}\Lambda$ is either projective–injective or simple.*
(2) *Any projective–injective module in* $\operatorname{mod}\Lambda$ *has length at most* 2.

By Proposition 4.4 and Corollary 4.5, we get the following

Corollary 4.6. *Let Λ be as in Corollary* 4.5 *and let $S \in \operatorname{mod}\Lambda$ be a simple module.*

(1) *If S is non-projective, then $0 \to \Omega^1 S \to P_0(S) \to S \to 0$ is an almost split sequence.*
(2) *If S is non-injective, then $0 \to S \to I^0(S) \to \Omega^{-1} S \to 0$ is an almost split sequence.*

Furthermore we get the following

Corollary 4.7. *Let Λ be as in Corollary* 4.5. *If Λ is connected, then the Auslander–Reiten quiver of Λ is the following*:

$$\begin{tikzcd}[column sep=small]
 & P(2) \arrow[dr] & & P(3) \arrow[dr] & & \cdots \arrow[dr] & & P(n) \arrow[dr] & & P(n+1) \arrow[dr] & \\
P(1) \arrow[ur] \arrow[rr, dotted, no head] & & S(2) \arrow[ur] \arrow[rr, dotted, no head] & & S(3) \arrow[ur] & & S(n-1) \arrow[ur] \arrow[rr, dotted, no head] & & S(n) \arrow[ur] \arrow[rr, dotted, no head] & & S(n+1)
\end{tikzcd}$$

where $P(i)$ and $S(i)$ are the ith projective module and simple module in $\operatorname{mod}\Lambda$ *respectively for any $1 \leqslant i \leqslant n+1$.*

Proof. It is not difficult to show that $S(n+1)$ is the unique simple module in $\operatorname{mod}\Lambda$ such that $\mathrm{pd}_\Lambda S(n+1) = n$ by Proposition 4.4. Then by Corollary 4.6 and Proposition 4.4 again, we get the assertion. $\square$

For an algebra Λ, we use $J(\Lambda)$ to denote the Jacobson radical of Λ. Now we are in a position to state the main result as follows.

Theorem 4.8. *Λ is an $(n-1)$-Auslander algebra with* $\mathrm{gl.dim}\,\Lambda = n$ *admitting a trivial maximal $(n-1)$-orthogonal subcategory of* $\operatorname{mod}\Lambda$ *if and only if it is Morita equivalent to a finite product of F and*
$$T_{n+1}(F)/J^2(T_{n+1}(F)) = \begin{pmatrix} F & F & 0 & \cdots & 0 & 0 \\ 0 & F & F & \cdots & 0 & 0 \\ 0 & 0 & F & \cdots & 0 & 0 \\ & \cdots & & & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & F & F \\ 0 & 0 & 0 & \cdots & 0 & F \end{pmatrix}_{(n+1)\times(n+1)},$$
where F is a division algebra and $T_{n+1}(F)$ is an $(n+1)\times(n+1)$ upper triangular matrix algebra over F.

Proof. It is straightforward to verify the sufficiency. In the following, we prove the necessity.

First, all $\mathrm{End}({}_\Lambda P(1)), \mathrm{End}({}_\Lambda P(2)), \ldots, \mathrm{End}({}_\Lambda P(n+1))$ are division algebras since the Auslander–Reiten quiver of Λ does not contain oriented cycles by Corollary 4.7. Moreover, they are mutually isomorphic since they are connected by arrows with trivial valuation $(1,1)$. Thus we get a division algebra $F := \mathrm{End}({}_\Lambda P(1)) \cong \mathrm{End}({}_\Lambda P(2)) \cong \cdots \cong \mathrm{End}({}_\Lambda P(n+1))$. Next, observe that $\mathrm{Hom}_\Lambda(P(i), P(i+1))$

is one-dimensional as left and right F-vector spaces, and that $\mathrm{Hom}_\Lambda(P(i), P(j)) = 0$ if $j \neq i, i+1$ by Corollary 4.7. Finally, since $J^2(\Lambda) = 0$ by Corollary 4.5, we get that Λ has the desired form. □

The algebra in Theorem 4.8 is $T_2^{(n)}(F)$ in Iyama's terminology (see [Iy6, Theorem 1.18]).

Recall from [AuRS] that Λ is called a *Nakayama algebra* if every indecomposable projective module and every indecomposable injective module in mod Λ have a unique composition series.

Corollary 4.9. *Let Λ be an $(n-1)$-Auslander algebra with* gl.dim $\Lambda = n$ *admitting a trivial maximal $(n-1)$-orthogonal subcategory of* mod Λ. *Then we have*:

(1) *Λ is a Nakayama algebra.*
(2) $\mathrm{pd}_\Lambda M + \mathrm{id}_\Lambda M = n$ *for any indecomposable non-projective–injective module $M \in$* mod Λ.
(3) $\mathrm{pd}_\Lambda M \leqslant n-1$ *or* $\mathrm{id}_\Lambda M \leqslant n-1$ *for any indecomposable module $M \in$* mod Λ.

Proof. It is straightforward to verify (1) by Theorem 4.8.

(2) By Corollary 4.5(1), any indecomposable non-projective–injective module in mod Λ is simple. Then it is not difficult to get the assertion by Corollary 4.7 or Theorem 4.8.

(3) Follows from (2) immediately. □

The following example illustrates that there exists a basic and connected $(n-1)$-Auslander algebra Λ with gl.dim $\Lambda = n$, which is a Nakayama algebra, but admits no maximal $(n-1)$-orthogonal subcategories of mod Λ.

Example 4.10. Let Λ be a finite-dimensional algebra over an algebraically closed field given by the quiver:

$$1 \xleftarrow{\beta_1} 2 \xleftarrow{\beta_2} 3 \xleftarrow{\beta_3} \cdots \xleftarrow{\beta_{2n-1}} 2n \xleftarrow{\beta_{2n}} 2n+1$$

modulo the ideal generated by $\{\beta_i\beta_{i+1} \mid 1 \leqslant i \leqslant 2n-1 \text{ but } i \neq n\}$. Then Λ is a basic and connected $(n-1)$-Auslander algebra with gl.dim $\Lambda = n$. By [ASS, Chapter V, Theorem 3.2], Λ is a Nakayama algebra. We use $P(i)$, $I(i)$ and $S(i)$ to denote the projective, injective and simple modules corresponding to the vertex i for any $1 \leqslant i \leqslant 2n+1$. Because $P(n+2) = I(n)$ is not simple, it follows from [ASS, Chapter IV, Proposition 3.11] that $0 \to P(n+1) \to S(n+1) \oplus P(n+2) \to I(n+1) \to 0$ is an almost split sequence. So $\mathrm{Ext}^1_\Lambda(I(n+1), P(n+1)) \neq 0$ and hence there does not exist a maximal j-orthogonal subcategory of mod Λ for any $j \geqslant 1$.

Recall from [HRS] that Λ is called *almost hereditary* if the following conditions are satisfied: (1) gl.dim $\Lambda \leqslant 2$; and (2) if $X \in$ mod Λ is indecomposable, then either $\mathrm{pd}_\Lambda X \leqslant 1$ or $\mathrm{id}_\Lambda X \leqslant 1$. Recall from [HRi] that Λ is called *tilted* if Λ is of the form $\Lambda = \mathrm{End}(T_\Gamma)$, where T_Γ is a tilting module over a hereditary Artinian algebra Γ.

Corollary 4.11. *Let Λ be an Auslander algebra with* gl.dim $\Lambda = 2$. *If Λ admits a trivial maximal* 1*-orthogonal subcategory of* mod Λ, *then Λ is a tilted algebra of finite representation type.*

Proof. By Corollary 4.9, Λ is an almost hereditary algebra of finite representation type. So Λ is tilted by [HRS, Chapter III, Corollary 3.6]. □

Remark 4.12. (1) Let Λ be an Auslander algebra (of finite representation type) with gl.dim $\Lambda = 2$. If Λ admits a non-trivial maximal 1-orthogonal subcategory of mod Λ (note: Iyama in [Iy6] constructed an example to illustrate that this may occur), then Λ is not almost hereditary because any maximal 1-orthogonal subcategory (if it exists) for an almost hereditary algebra is trivial by [HuZ, Theorem 3.15]. So Λ is not tilted.

(2) In the statement of Corollary 4.11, the conditions "Λ is an Auslander algebra" and "Λ is a tilted algebra of finite representation type" cannot be exchanged. For example, let Λ be a finite-dimensional algebra given by the quiver:

$$1 \xleftarrow{\alpha_1} 2 \xleftarrow{\alpha_2} 3 \xleftarrow{\alpha_3} 4 \xleftarrow{\alpha_4} 5$$

modulo the ideal generated by $\{\alpha_1\alpha_2\alpha_3\alpha_4\}$. Then Λ is a tilted algebra of finite representation type (cf. [ASS, p. 323]), and Λ admits a trivial maximal 1-orthogonal subcategory $\mathrm{add}_\Lambda \bigoplus_{i=1}^{5} P(i) \oplus I(3) \oplus I(4) \oplus I(5)$ of mod Λ. However, Λ is not an Auslander algebra because $\mathrm{pd}_\Lambda I^1(\Lambda) = 2$.

5. Non-trivial maximal 1-orthogonal subcategories

In this section, based on [HuZ, Corollary 3.12], we will further give some necessary condition for Auslander algebras with global dimension 2 admitting a non-trivial maximal 1-orthogonal subcategory in terms of the homological properties of simple modules.

Lemma 5.1. *Let Λ be an Auslander algebra with* $\mathrm{gl.dim}\,\Lambda = 2$ *and let $S \in \mathrm{mod}\,\Lambda$ be a simple module with* $\mathrm{id}_\Lambda S = 2$. *Then $I^2(S)$ is indecomposable and $I^0(S) \ncong I^1(S)$.*

Proof. By Lemma 3.5, there exists a simple module $S' \in \mathrm{mod}\ \Lambda$ such that $\mathrm{pd}_\Lambda S' = 2$ and $\mathbb{D}\,\mathrm{Ext}^2_\Lambda(S', \Lambda) = S$. From the minimal projective resolution of S', we get an exact sequence:

$$0 \to P_0(S')^* \to P_1(S')^* \to P_2(S')^* \to \mathrm{Ext}^2_\Lambda(S', \Lambda) \to 0,$$

which is a minimal projective resolution of $\mathrm{Ext}^2_\Lambda(S', \Lambda)$ by [M, Proposition 4.2]. Then applying the functor $\mathbb{D}$, we get a minimal injective resolution of $S = \mathbb{D}\,\mathrm{Ext}^2_\Lambda(S', \Lambda)$:

$$0 \to S \to \mathbb{D}P_2(S')^* \to \mathbb{D}P_1(S')^* \to \mathbb{D}P_0(S')^* \to 0.$$

It follows that $I^2(S) \cong \mathbb{D}P_0(S')^*$, $I^1(S) \cong \mathbb{D}P_1(S')^*$ and $I^0(S) \cong \mathbb{D}P_2(S')^*$. On the other hand, from the minimal projective resolution of S':

$$0 \to P_2(S') \to P_1(S') \to P_0(S') \to S' \to 0,$$

we know that $P_0(S')$ is indecomposable and $P_2(S') \ncong P_1(S')$. So the assertion follows. $\square$

Proposition 5.2. *Let Λ be an Auslander algebra with* $\mathrm{gl.dim}\,\Lambda = 2$. *If Λ admits a non-trivial maximal 1-orthogonal subcategory of* mod Λ, *then we have*:

(1) *There exists a simple module in* mod Λ *with both projective and injective dimensions* 2.
(2) *There exist at least two non-injective simple modules in* mod Λ *with projective dimension* 2.

Proof. (1) It follows from [HuZ, Corollary 3.12].

(2) By (1), there exists a non-injective simple module in mod Λ with projective dimension 2. If the non-injective simple module in mod Λ with projective dimension 2 is unique (say S), then $\mathrm{id}_\Lambda S = 2$ by (1). Since $I^0(S)$ and $I^2(S)$ are indecomposable by Lemma 5.1, $\mathrm{grade}\, I^0(S) = \mathrm{grade}\, I^2(S) = 2$ by Lemma 3.10. Put $K = \mathrm{Coker}(S \hookrightarrow I^0(S))$. Then $\mathrm{grade}\, K = 2$ by Lemma 3.9 and so $\mathrm{grade}\, I^1(S) = 2$. We claim that $I^0(S)$ is isomorphic to a direct summand of $I^1(S)$. Otherwise, since S is the unique non-injective simple module with projective dimension 2, any non-zero indecomposable direct summand of $I^1(S)$ is simple by Lemma 3.8(2). So $I^1(S)$ is semisimple and hence K is injective, which contradicts the fact that $\mathrm{id}_\Lambda S = 2$. The claim is proved.

Notice that $I^2(S)$ is indecomposable and $\mathrm{pd}_\Lambda I^2(S) = 2$, so $I^2(S) \cong I^0(S)$ or $I^2(S) \cong S'$ for some simple module $S' \in \mathrm{mod}\ \Lambda$ such that $S \ncong S'$ and $\mathrm{pd}_\Lambda S' = 2$. In the latter case, we have that $\text{Ł}(I^0(S)) = \text{Ł}(I^1(S))$. Since $I^0(S)$ is isomorphic to a direct summand of $I^1(S)$ by the above argument, $I^0(S) \cong I^1(S)$, which is a contradiction by Lemma 5.1.

Because Λ is an Auslander algebra and $\mathrm{pd}_{\Lambda^{op}} \mathbb{D}S = 2$, it follows from Lemma 3.10 that $\mathrm{grade}\, \mathbb{D}S = 2$. Then $\mathrm{Ext}^1_\Lambda(I, S) \cong \mathrm{Ext}^1_{\Lambda^{op}}(\mathbb{D}S, \mathbb{D}I) = 0$ for any injective module $I \in \mathrm{mod}\ \Lambda$. Moreover, $S \hookrightarrow I^0(S)$ is left minimal, thus K has no injective direct summands by Lemma 2.5 and therefore K is indecomposable by Lemmas 5.1 and 2.1. It follows from Lemma 2.3 that $I^1(S) \to I^2(S)$ is right minimal. So, if $I^2(S) \cong I^0(S)$, then $I^1(S)$ has no simple direct summand S'' such that $S'' \ncong S$ and $\mathrm{pd}_\Lambda S'' = 2$. It yields that $I^1(S) \cong [I^0(S)]^t$ for some $t \geqslant 1$ and $2\text{Ł}(I^0(S)) = t\text{Ł}(I^0(S)) + 1$. It implies that $t = 1$ and $I^0(S) \cong I^1(S)$, which is a contradiction by Lemma 5.1. The proof is finished. □

We end this section with some examples to illustrate Proposition 5.2.

The following example shows that there exists an Auslander algebra Λ with $\mathrm{gl.dim}\, \Lambda = 2$ satisfying the condition (1) in Proposition 5.2, but not satisfying the condition (2) in this proposition.

Example 5.3. When $n = 2$, the algebra Λ in Example 4.10 satisfies the conditions:

(1) Λ is an Auslander algebra with $\mathrm{gl.dim}\, \Lambda = 2$.
(2) All simple modules in mod Λ with projective dimension 2 are $S(3)$ and $S(5)$.
(3) $\mathrm{id}_\Lambda S(3) = 2$ and $S(5)$ is injective.

Then by Lemma 3.7 and Proposition 5.2(2), there does not exist any maximal 1-orthogonal subcategory of mod Λ.

The following example shows that there exists an Auslander algebra Λ with $\mathrm{gl.dim}\, \Lambda = 2$ satisfying the condition (2) in Proposition 5.2, but not satisfying the condition (1) in this proposition.

Example 5.4. Let Λ be a finite-dimensional algebra given by the quiver:

$$\begin{tikzcd}
6 \arrow[r, "\alpha"] \arrow[d, "\gamma"] & 4 \arrow[d, "\beta"] & \\
5 \arrow[r, "\delta"] & 3 \arrow[r, "\lambda"] \arrow[d, "\mu"] & 1 \\
& 2 &
\end{tikzcd}$$

modulo the ideal generated by $\{\beta\alpha - \delta\gamma, \mu\delta, \lambda\beta\}$. Then we have

(1) Λ is an Auslander algebra and an almost hereditary algebra with $\mathrm{gl.dim}\, \Lambda = 2$.
(2) All simple modules in mod Λ with projective dimension 2 are $S(4)$, $S(5)$ and $S(6)$.
(3) $\mathrm{id}_\Lambda S(4) = \mathrm{id}_\Lambda S(5) = 1$ and $S(6)$ is injective.

Then by Lemma 3.7 and Proposition 5.2(1), there does not exist any maximal 1-orthogonal subcategory of mod Λ.

By Examples 5.3 and 5.4, we have that the conditions (1) and (2) in Proposition 5.2 are independent.

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